Fuzzy Modeling

- **Application example**
- **Mathematical models**
- **Steps to build fuzzy systems**

Economic models

- Models relate variables of interest
- Often, equilibrium solutions are sought
- Relation 1, variable 1 – variable 2
- Relation 2, variable 1 – variable 2
- Relations between variables are usually assumed to be linear
- The equilibrium solution satisfies both relations

Supply-demand relationship

- Relate price to the supply quantity

$$Q = aP + b, \quad a > 0$$

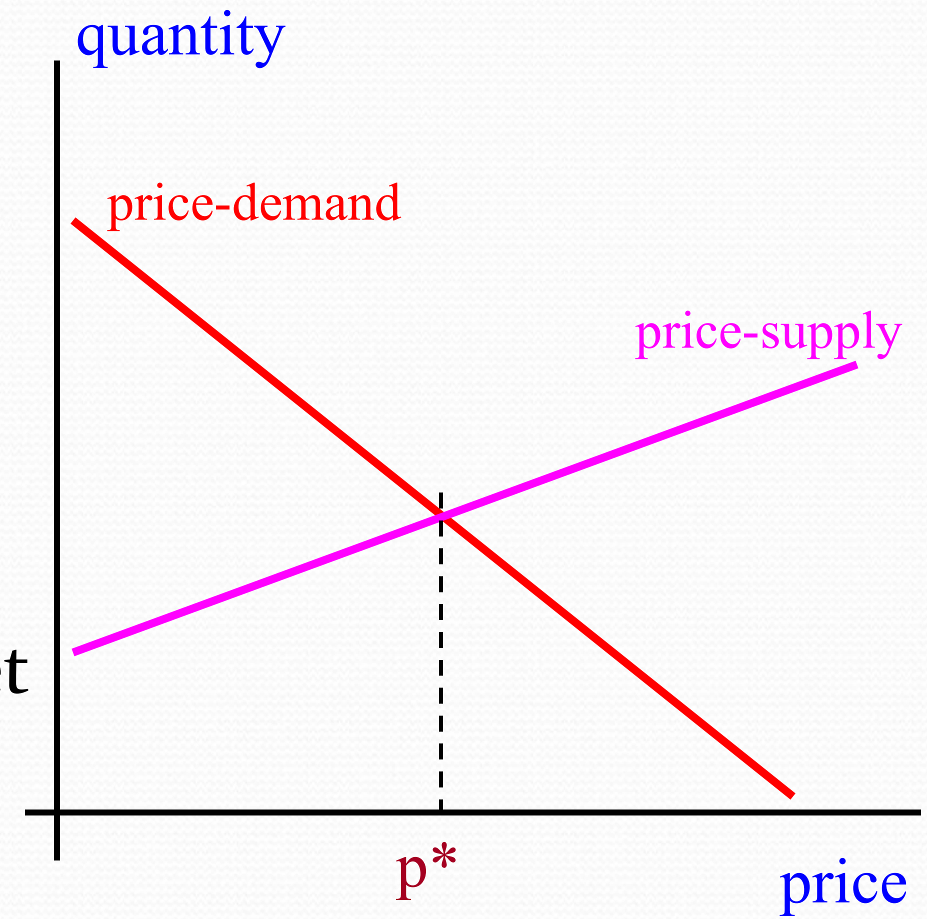
- Relate price to the demand quantity

$$Q = cP + d, \quad c < 0$$

- Solve simultaneous set of equations

$$Q = aP + b, \quad a > 0$$

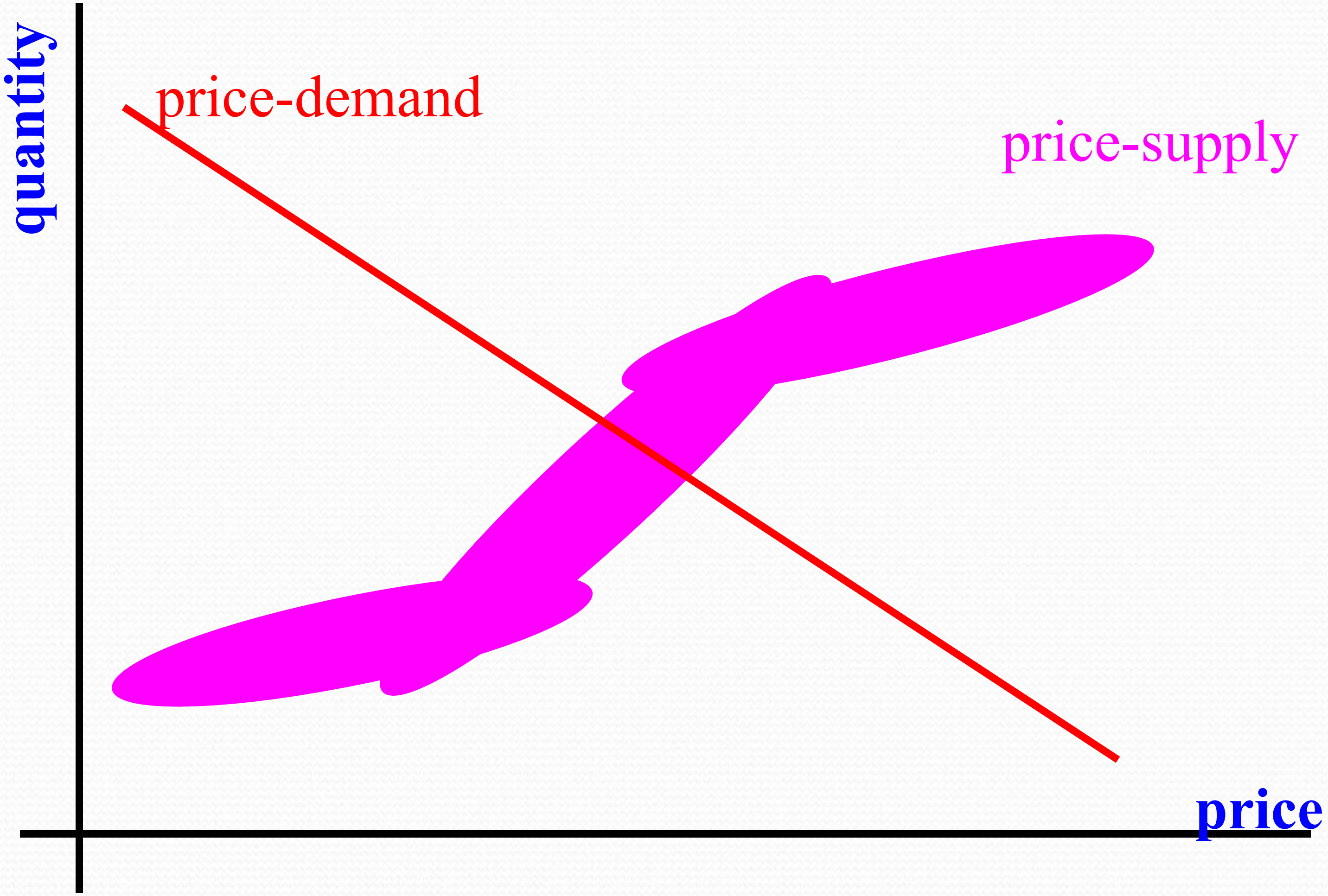
$$Q = cP + d, \quad c < 0$$



Price-supply relationship

- For very low prices, there is little incentive to produce any goods, as price is less than cost
- As the price moves up, economic incentive appears for production, and an increase in price will cause an increase in production
- Finally, there is a saturation region, as no further increase is possible

Fuzzy supply, crisp demand relation



Zero-order Takagi-Sugeno model

Fuzzy price-supply relation

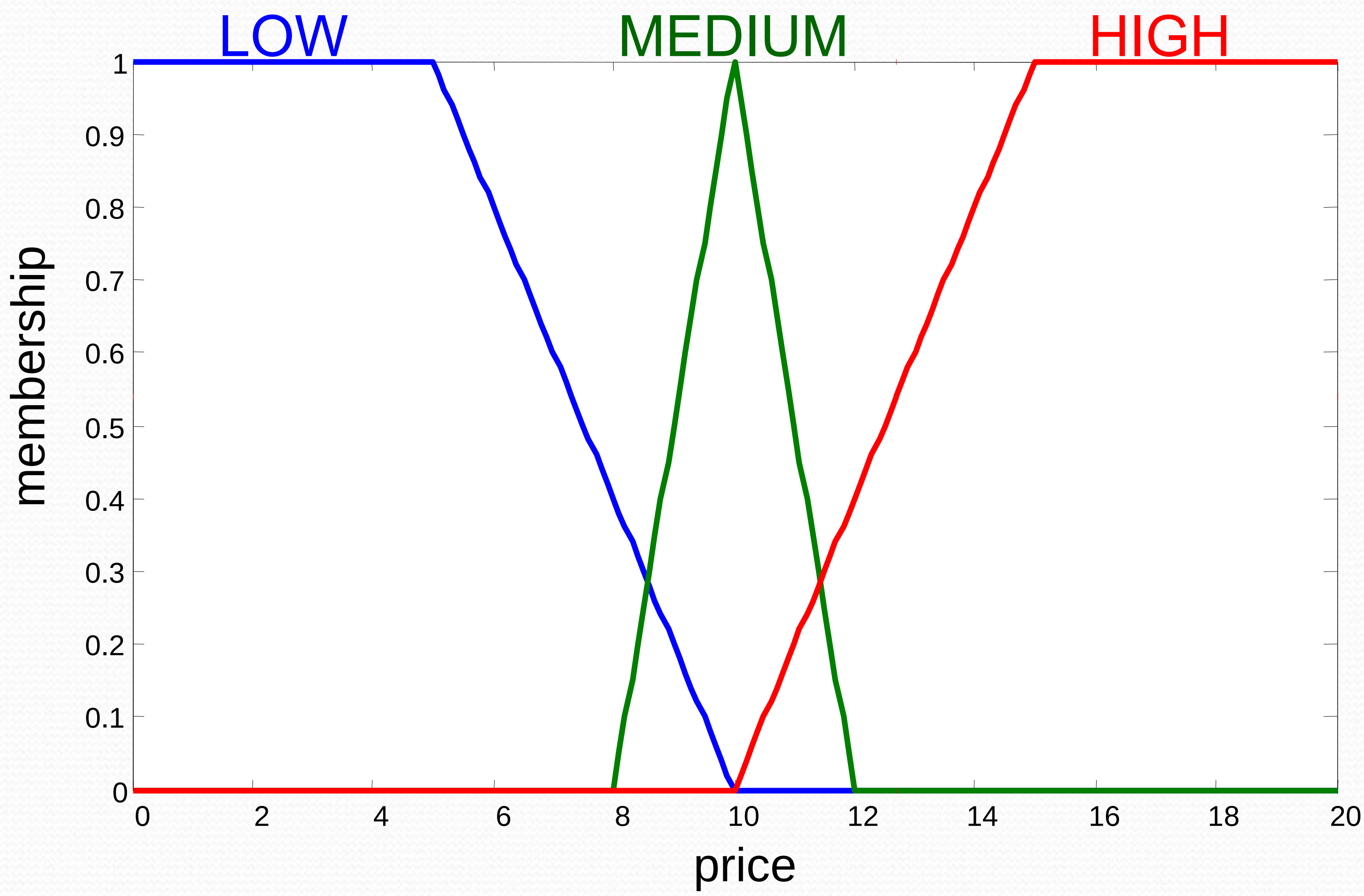
- If price is LOW then supply is $g_1(p)$
- If price is MEDIUM then supply is $g_2(p)$
- If price is HIGH then supply is $g_3(p)$

$g_1(p) = 8 \qquad g_2(p) = 22 \qquad g_3(p) = 30$

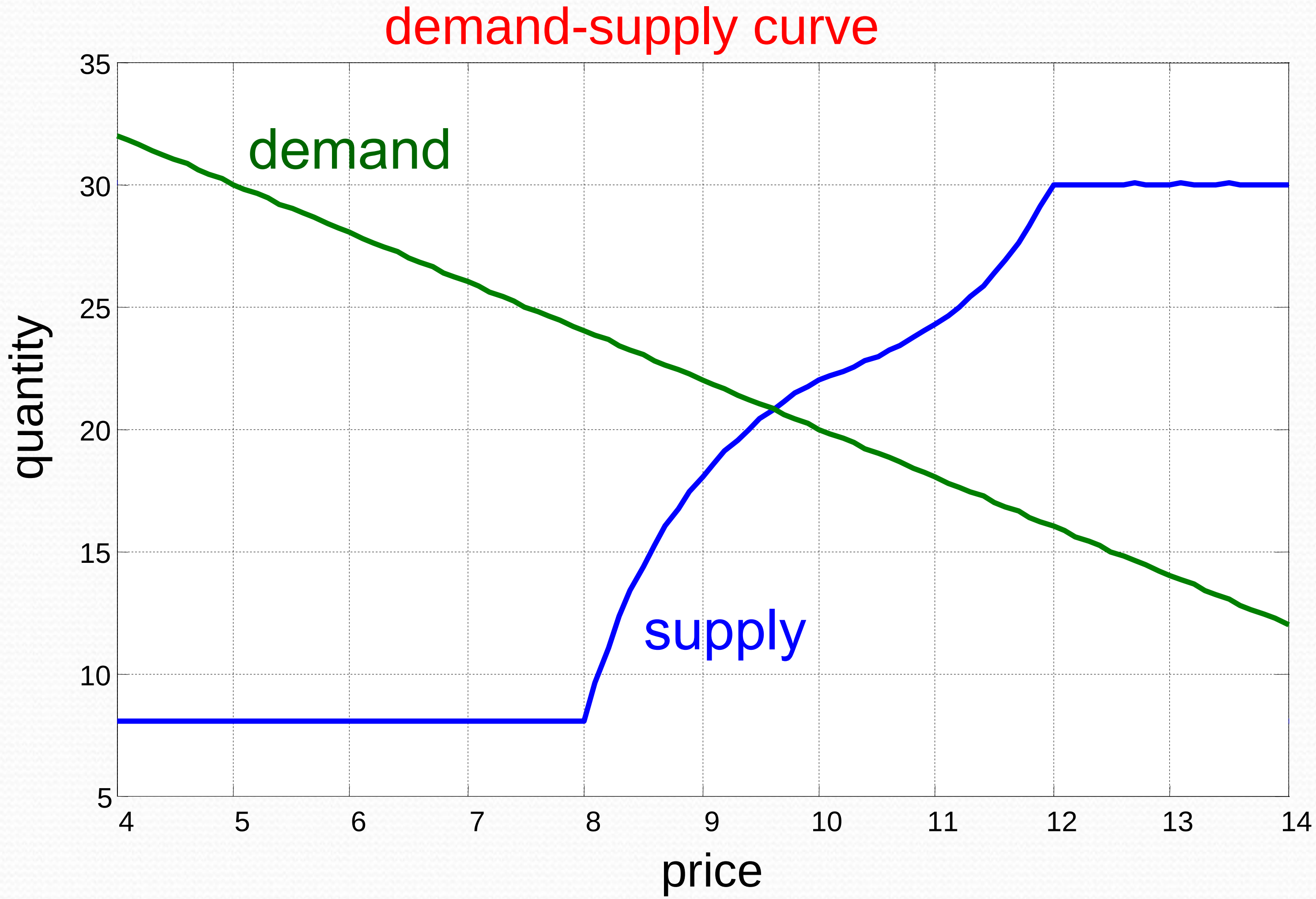
Crisp price-demand relation

demand = 40 – 2p

Membership functions



Simultaneous solution



Analytical solution

- Consider piecewise linear membership functions
- Use a zero-order Takagi-Sugeno model
- Divide price range into a number of regions within which a solution is sought by solving the equation

Degree of fulfillment

$$cp + d = \underbrace{\frac{\sum_{j=1}^{N} A_j(p)\, g_j(p)}{\sum_{j=1}^{N} A_j(p)}}_{\text{TS supply model}}$$

Rule consequent

Example

$$A_1(p) = \begin{cases} 1 & \text{for } 0 \le p < 5 \\ -\frac{1}{5}p + 2 & \text{for } 5 \le p < 10 \\ 0 & \text{for } p \ge 10 \end{cases}$$

$$A_2(p) = \begin{cases} 0 & \text{for p} < 8 \\ \frac{1}{2}p - 4 & \text{for } 8 \le \text{p} < 10 \\ -\frac{1}{2}p + 6 & \text{for } 10 \le \text{p} < 12 \\ 0 & \text{for p} \ge 12 \end{cases}$$

$$A_3(p) = \begin{cases} 0 & \text{for } p < 10 \\ \frac{1}{5}p - 2 & \text{for } 10 \le p < 15 \\ 1 & \text{for } p \ge 15 \end{cases}$$

- Solution is given by

$$40 - 2p = \\ = \frac{8A_1(p) + 22A_2(p) + 30A_3(p)}{A_1(p) + A_2(p) + A_3(p)}$$

- For $p \in [0,5]$

$$A_1(p) = 1, \quad A_2(p) = A_3(p) = 0$$

$$40 - 2p = 8 \quad \Rightarrow \quad p = 16$$

$$\text{solution is } \varnothing$$

Example

- For p ∈ [8,10]

$$A_1(p) = -\frac{1}{5}p + 2$$

$$A_2(p) = \frac{1}{2}p - 4$$

$$A_3(p) = 0$$

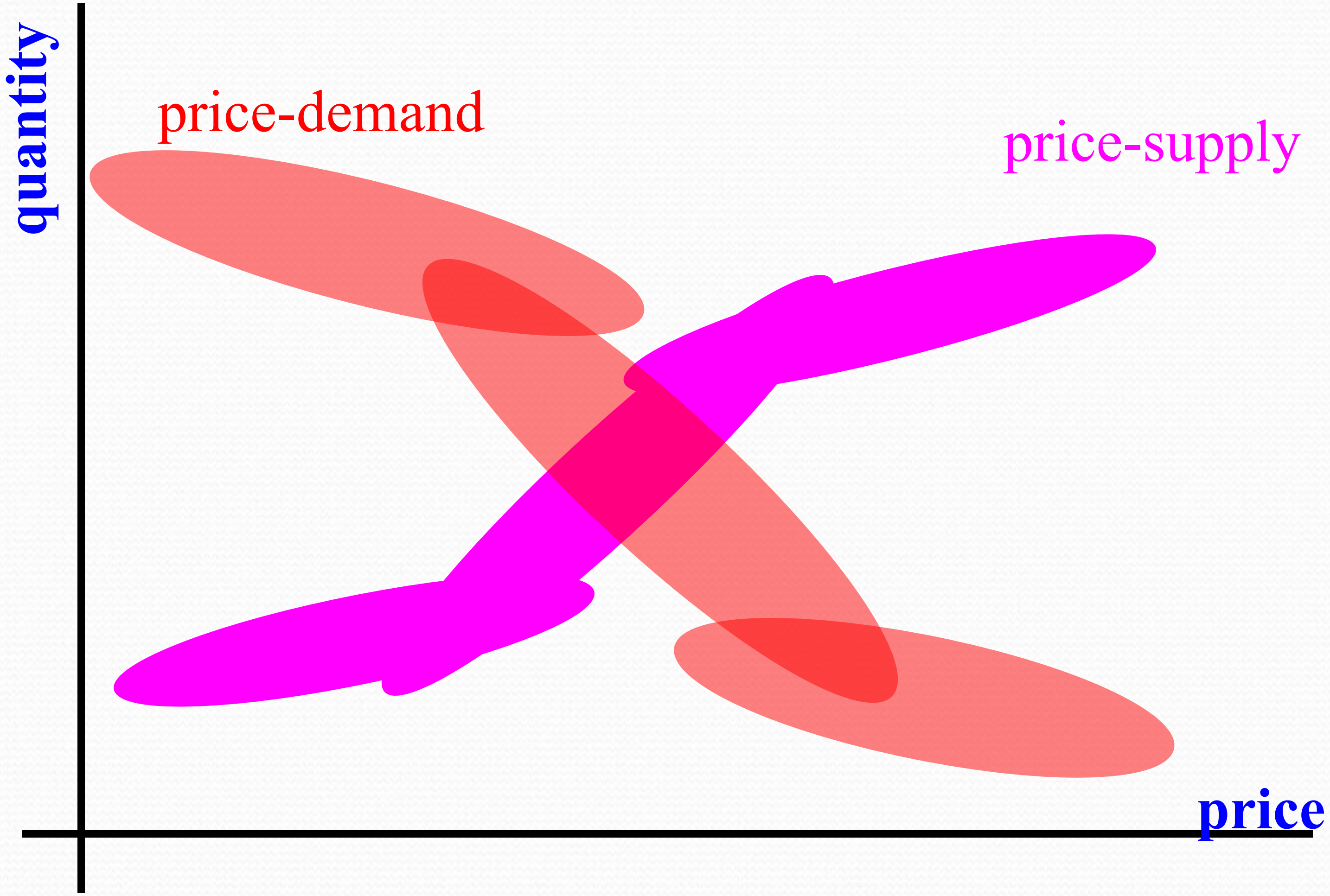
$$40 - 2p = \frac{8\left(-\frac{1}{5}p + 2\right) + 22\left(\frac{1}{2}p - 4\right)}{\left(-\frac{1}{5}p + 2\right) + \left(\frac{1}{2}p - 4\right)}$$

$$3p^2 - 33p + 40 = 0$$

$$p_1 = 9.6 \quad \text{or} \quad p_2 = 1.3$$

Hence, p = 9.6 is the solution

Solution for two fuzzy relations



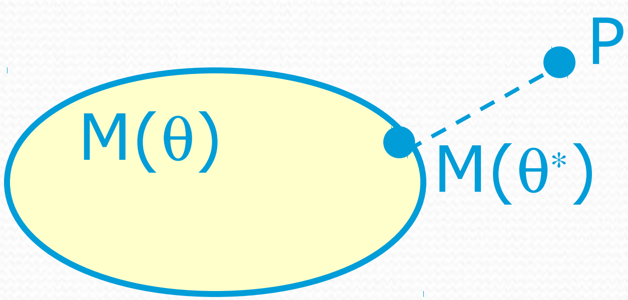
Mathematical models

- A representation of reality
- However, appropriateness of a model is coupled to one's goals
- A model consists of
  - Structure: variables, inputs, outputs and types of relations amongst them
  - Parameters: free variables after a structure is selected
  - Search method: method with which the optimal parameters are identified

Modeling and optimization

- Given a process P
- Consider a class M of models parameterized by θ
- Optimal model by

$$\theta^* = \arg\min_{\theta} \| P - M(\theta) \|$$



Example

- A model with a linear structure
- Two inputs
- One output
- Three parameters

$$y = a_1 x_1 + a_2 x_2 + a_0$$

- LMS for optimizing sum of squared errors

$$J = \sum_{k=1}^{N} (y_k - \hat{y}_k)^2$$

Why build models?

- To predict a system's behaviour
- To explain interactions and relationships between inputs and outputs
- To simulate a system and design controllers

What needs to be determined?

Fuzzy inference systems:

- Rule base (structure)
  (rule mapping)
- Definition of membership functions
  (inputs and outputs)
- Estimation of consequent parameters
  (for Sugeno systems)

How to obtain fuzzy models?

- *Expert knowledge driven*
  - Initialize FS using expert knowledge
  - Optimize parameters with expert knowledge
- *Data driven*
  - Two stage:
    - partition input/output space (linguistic variables, values)
    - learn rules
  - Iterative refinement procedure (e.g., neuro-fuzzy approach)
- *Combinations of above*
  - data-driven estimation of optimal parameters after a suitable expert-driven initialization
  - ...

Basic steps to build fuzzy systems

- Determine the relevant/available input and output variables/features
- Determine suitable universe of discourse and a term set for the variables
- Define membership functions for linguistic terms
- Determine fuzzy rules for the rule base
- Determine model choices and parameters (including inference operators)
- Tune the system (deep structure)

Rule induction

- Location of kernels
  membership function positions
- Extension of kernels
  spread of membership functions
- Shape of kernels
  type of membership functions
- Consequents associated with kernels
  estimation of consequent parameters

Rulebase optimization

Interpretability influenced by

- Input and output variables
  number, relevance and interpretation
- Membership functions
  number and label meanings
- Rulebase
  number of rules and interpretation; possibility for incomplete rules

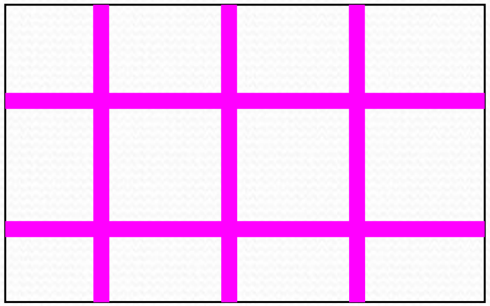
Interplay with expert knowledge

- When designing FIS, use expert knowledge whenever available
- Multiple strategies possible, e.g.
  - initialize a rulebase with expert knowledge and optimize with data
  - design a rulebase from data and optimize with expert knowledge
  - use experts to validate final rulebase
  - use experts to provide boundary conditions for rulebase optimization

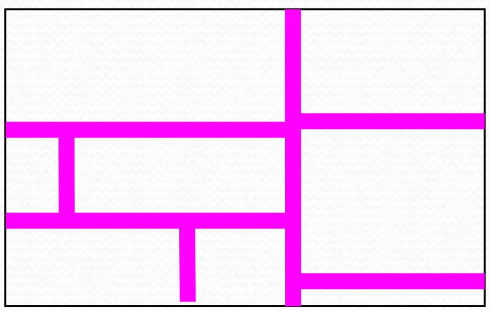
Input space partitioning

- Rule induction partitions the input space into number of fuzzy (overlapping) regions
- The type of partition generated depends on chosen algorithm
- Optimization modifies membership parameters to define a partition that minimizes output error

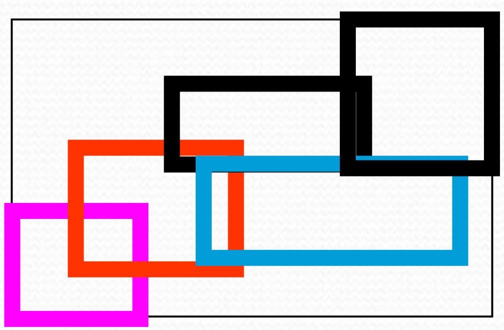
Input space partitioning



Grid partition



Tree partition



Scatter partition

Finding partitions

- Clustering algorithms
- Optimizing (e.g. with a genetic algorithm, neural networks)
- Decision trees (tree partition – see data mining) based on an information criterion

Fuzzy Logic vs Probability Theory

- Probability theory $\subseteq$Fuzzy Logic?
  - Product T-norm = probability of independent events
  - Membership with area 1 = distribution
- But:
  - (Standard) Fuzzy logic does not have conditional probability
  - (Standard) Fuzzy logic does not have a Bayes' rule $P(A|B)P(B)=P(B|A)P(A)$
  - Fuzzy logic does not satisfy the "3 axioms of probability theory" → As a representation of belief it is not "coherent" in the eyes of statisticians

Fuzzy Logic vs Probability Theory

- Assume that these properties are desirable:
  - Representation of plausibility by real numbers
  - Reflection of common sense
    if a number is higher, this reflects higher plausibility
  - Consistency
    if multiple lines of reasoning exist, all will lead to the same conclusion